

USC-FT-9-01  
hep-th/0107079

# Math and Physics<sup>1</sup>

JOSÉ M. F. LABASTIDA

*Departamento de Física de Partículas  
Universidade de Santiago de Compostela  
15706 Santiago de Compostela, Spain  
e-mail: labasti@fpaxpl.usc.es*

## Abstract

I present a brief review on some of the recent developments in topological quantum field theory. These include topological string theory, topological Yang-Mills theory and Chern-Simons gauge theory. It is emphasized how the application of different field and string theory methods has led to important progress, opening entirely new points of view in the context of Gromov-Witten invariants, Donaldson invariants, and quantum-group invariants for knots and links.

## 1 Introduction

The last two decades constituted a very fruitful time for theoretical physics. During this period the use of geometrical and topological methods have been particularly intense, leading to a new type of relation between physics and mathematics. Beginning in the eighties, we have witnessed how the most advanced developments in theoretical physics have led to new results in mathematics. A new type of relation emerged which is unprecedented in history. Mathematics and physics enjoyed a happy marriage during hundreds of years and evolved together up to the end of the XIX<sup>th</sup> century. Official divorce took place at the beginning of the XX<sup>th</sup> century when abstraction started to play

---

<sup>1</sup>Talk delivered at the special seminar in honor of F. J. Ynduráin held at Sitges, Spain, on February 9, 2001

a fundamental role in mathematics. During their happy years new fields in mathematics were often created because they were needed by physics. In the twentieth century, however, mathematics was always ready for physics needs. This was the case, for example, of general relativity and quantum mechanics. One of the members of the couple did not need the other anymore and their relation cooled down.

The eighties constituted a period of reconciliation. Physics had the courage to tell its old partner were to look at and this did it quite successfully. Quantum field theory and string theory started to generate new mathematics, establishing a new type of relation which hopefully will last for many years. The most remarkable aspect of this new liaison is that the emerging new mathematics were not produced because they were needed by physics but because they could lead to important breakthroughs in mathematics. Topology was the field of mathematics that was particularly involved in these developments. On the physics side, the quantum aspects of field and string theory were the responsible for the establishment of the new connections in mathematics. All these developments have led to the study of new types of theories now known as topological field theories and topological string theories.

The motivation for the study of these theories is not only mathematical. There are reasons that make them also interesting from a physics point of view. First, they are simpler than theories related to the real world. This allows to solve them exactly in some cases, generating an important knowledge on the structure of field and string theory. Second, these theories could be used as starting points to solve more realistic theories. Third, they constitute an excellent framework to test physics arguments which are not firmly supported from a mathematical point of view.

The third motivation is indeed related to the cyclic nature of the new relation between physics and mathematics. Results in field and string theory which use arguments based on functional integrals (arguments of quantum origin) can not be considered in general as firm arguments from a mathematical perspective. Those results have to be considered as predictions in mathematics made by physics. Once the predictions are proposed, mathematicians have to provide the corresponding rigorous proofs. In a certain sense, mathematicians play the role of experimental physicists when theoretical physicists make predictions that have to be tested in the laboratories. Needless to say that the predictions made by topological field and string theory are much cheaper to test than the ones made by ordinary theories.

The analogy with experimental physics can be pursued further. Often, from the test of theoretical predictions in laboratories, new phenomena is observed that leads to new theoretical studies. Similarly, from the rigorous proofs of the predictions made from topological field theory and string theory new facts are encountered that, in turn, lead to further study in physics. These, eventually, could lead to additional new results in mathematics, establishing a cyclic structure as in the case of experimental physics.

A particularly good example to illustrate this cyclic behavior is Morse-Witten theory. In 1982 Witten studied supersymmetric quantum mechanics and supersymmetric sigma models making a prediction on the existence of an interesting generalization of Morse theory [1]. This prediction was later studied by A. Floer, establishing its fundamentals from a mathematical perspective [2]. The theory was later extended by Floer himself to other contexts, motivating further studies by Witten which, after some pressure by M. Atiyah [3], culminated in the formulation of Donaldson theory [4] in a field theory framework in 1988 [5].

The developments of the type discussed above have been taking place in field theories defined in 2, 3 and 4 dimensions, and in the context of string theory. In this article I will present a brief review of these developments emphasizing on the recent ones in the context of knot theory. These are particularly important because they show how useful field and string theory are to unravel the many faces of the theory of knot and link invariants. The formalism leads to relations which otherwise would have been rather hard to discover.

The cases that will be considered in this paper have two common features. In all of them, first, a particular problem in topology is formulated in terms of field or string theory. In particular, we will be dealing with Gromov-Witten invariants, Donaldson invariants and quantum-group invariants. For all these invariants, topological field theory or topological string theory provide a representation in terms of vacuum expectation values of some “physical observables”. Second, field and string theory methods are used to express these vacuum expectation values in alternative forms, leading to expressions that involve new integer invariants. In the case of Donaldson invariants these new integer invariants are the celebrated Seiberg-Witten invariants. Similarly, in the case of Gromov-Witten invariants one finds that these can be resumed and expressed in terms of simpler integer invariants.

For quantum-group invariants progress evolved in two steps. First, the quantum-group polynomial invariants, whose coefficients are integer num-

bers, are expressed, after a string theory reformulation of Chern-Simons gauge theory, in terms of new non-integer invariants which are a particular analog of Gromov-Witten invariants for Riemann surfaces with boundaries. In the second step these non-integer coefficients are resumed in terms of new integer invariants which are related by a simple reformulation to the integer coefficients of quantum-group invariants. These new invariants can be interpreted in terms of enumerative geometry and therefore the formalism assigns a geometrical meaning to the integer coefficients of the quantum-group polynomial invariants, a question that remained open since their formulation.

The organization of the paper is as follows. In section 2, I briefly define quantum field theory and I present a brief review of these theories in 2 and 4 dimensions, dealing with Gromov-Witten invariants, and Donaldson and Seiberg-Witten invariants, respectively. In section 3, I review Chern-Simons gauge theory and its role in the theory of knot and link invariants, making a special emphasis in its latest developments related to string theory.

## 2 Topological quantum field theory in two and four dimensions

Topological quantum field theories are quantum field theories whose correlation functions lead to topological invariants. They can be constructed in two ways, leading to two types of theories: Schwarz and Witten types. In the first case one considers theories whose action is manifestly independent of the metric. Then, if the observables do not involve the metric either, correlators lead to topological invariants (if there are no metric dependences induced upon quantization). An example of Schwarz-type theory is Chern-Simons gauge theory.

In Witten-type theories, also known as theories of cohomological type, though the action manifestly depends on the metric, a symmetry prevents the correlators of being metric dependent. These theories possess a supersymmetric counterpart and their characteristic symmetry is indeed related to supersymmetry. Topological quantum field theories of this type are twisted versions of ordinary  $\mathcal{N} = 2$  supersymmetric theories. Under the twist the Lorentz transformations of some of the fields are modified and, though on flat space both theories are the same, on curved space are different. On curved space only one of the supersymmetry transformations survives as a

symmetry. This symmetry is a nilpotent scalar quantity  $Q$ . The operators whose correlators lead to topological invariants, or “physical observables”, are the elements of the cohomology generated by  $Q$ . In these theories the energy-momentum tensor is  $Q$ -exact and thus the elements of the cohomology of  $Q$  lead to quantities which are metric independent (modulo potential anomalies). In these theories the functional integral is typically localized on field configurations which are  $Q$ -invariant, leading to a moduli problem. The two classical examples of these types of topological quantum field theories are topological sigma models in two dimensions [6] and topological Yang-Mills theory in four dimensions [5].

Topological sigma models are obtained after twisting  $\mathcal{N} = 2$  supersymmetric sigma models. The twisting can be done in two different ways leading to two types of models, A and B [7]. Their existence is related to mirror symmetry. Only type-A models will be described in what follows. These models can be defined on an arbitrary almost complex manifold, though typically they are considered on Kähler manifolds. The theory involves maps from two-dimensional Riemann surfaces  $\Sigma$  to target spaces  $X$ , together with fermionic degrees of freedom on  $\Sigma$  which are mapped to tangent vectors on  $X$ . The functional integral of the resulting theory is localized on holomorphic maps, defining the corresponding moduli space. The  $Q$ -cohomology provides the set of physical observables. They can be integrated as classes over the moduli space leading to topological invariants.

Topological sigma models keep fixed the complex structure of the Riemann manifold  $\Sigma$ . Motivated by string theory one also considers the situation in which one integrates over complex structures. In this case one ends working with holomorphic maps in the entire moduli space of curves. The resulting theories are called topological strings.

I will review now a particular example of topological string theory which, besides being very interesting from the point of view of physics and mathematics, it will be very useful in the discussion presented in the next section. Let us consider topological strings choosing  $X$  to be a Calabi-Yau threefold. In this case the virtual dimension of the moduli space of holomorphic maps turns out to be zero. Two situations can occur: the space is given by a number of points or the space involves a moduli and possesses a bundle of the same dimension as the tangent bundle. In the first case, topological strings count the number of points weighted by the exponential of the area of the holomorphic map (the pull back of the Kähler form integrated over the surface) times  $x^{2g-2}$  where  $x$  is the string coupling constant and  $g$  is the genus

of  $\Sigma$ . In the second case one computes the top Chern class of the appropriate bundles (properly defined), again weighted by the same factor. In both cases one can classify the contributions according to the cohomology class  $\beta$  on  $X$  in which the image of the holomorphic map is contained. The sum of the numbers obtained for each  $\beta$  and fixed  $g$  are known as Gromov-Witten invariants,  $N_g^\beta$ . The topological string contribution takes the form:

$$\sum_{g \geq 0} x^{2g-2} \left( \sum_{\beta \in H_2(X, \mathbb{Z})} N_g^\beta e^{\int_\beta \omega} \right) \quad (1)$$

where  $\omega$  is the Kähler class of the Calabi-Yau manifold. In general, the numbers  $N_g^\beta$  are rational numbers.

The discussion has shown how Gromov-Witten invariants can be interpreted in terms of string theory. One could think that this is just a nice observation and that no new insight on these invariants could be obtained from this formulation. The situation turns out to be quite the opposite. Once a string formulation has been obtained the whole machinery of string theory is at our disposal. One should look to new ways to compute the quantity (1), where Gromov-Witten invariants are packed. The hope is that, if this is possible, the new emerging picture will provide new insights on these invariants. This is indeed what occurred in the last three years. It turns out that the quantity (1) can be obtained from an alternative point of view in which the embedded Riemann surfaces are regarded as D-branes [8]. The outcome of this approach is that the Gromov-Witten invariants can be written in terms of other invariants which are integers and that posses a geometrical interpretation. To be more specific, the quantity (1) takes the form:

$$\sum_{\substack{g \geq 0 \\ \beta \in H_2(X, \mathbb{Z})}} \sum_{d > 0} n_g^\beta \frac{1}{d} (2 \sin(\frac{dx}{2}))^{2g-2} e^{d \int_\beta \omega} \quad (2)$$

where  $n_g^\beta$  are the new *integer* invariants. This prediction has been verified in all the cases in which it has been tested [9]. A similar structure will be found in the next section in the context of knot theory in the large- $N$  limit.

I will now briefly review topological Yang-Mills theory in four dimensions. As the topological sigma models, this theory is constructed by twisting  $\mathcal{N} = 2$  supersymmetric Yang-Mills theory. This process modifies the spin content

of the fields of the theory, leading to a new nonequivalent theory on curved manifolds. Again, out of the full  $\mathcal{N} = 2$  supersymmetry transformations, a nilpotent scalar symmetry is preserved that guaranties the topological character of the theory. The corresponding cohomology fixes the physical observables. It turns out that in this theory the contributions from the functional integral are localized on the moduli space of instantons. The correlators turn out to be integrals of appropriate forms on this moduli space, leading to quantities that are identified with Donaldson invariants.

To be more specific, let us consider the case in which the gauge group is  $SU(2)$ , and the four manifold  $X$  is simply connected and has  $b_2^+ > 1$  (the case  $b_2^+ = 1$  is anomalous). In this situation there are  $1 + b_2$  physical observables,  $\mathcal{O}$  and  $I(\Sigma_a)$ ,  $a = 1, \dots, b_2$ , where  $\Sigma_a$  is a basis of  $H_2(X)$ . After packing these observables in a generating functional, topological Yang-Mills theory leads to the computation of the following functional integral:

$$\left\langle \exp \left( \sum_a \alpha_a I(\Sigma_a) + \lambda \mathcal{O} \right) \right\rangle, \quad (3)$$

where  $\lambda$  and  $\alpha_a$ ,  $a = 1, \dots, b_2$ , are parameters. In computing this quantity one can argue that the contribution is localized on the moduli space of instantons configurations and one ends, after taking into account the selection rule dictated by the dimensionality of the moduli space, with integrations over the moduli space of the selected forms. The resulting quantities are precisely Donaldson invariants.

As in the case of topological sigma models one could be tempted to argue that the observation leading to a field-theoretical interpretation of Donaldson invariants does not provide any new insight. It would be a mistake to do so. Once a field theory formulation is available one has at his disposal a huge machinery which could lead, on the one hand, to further generalizations of the theory and, on the other hand, to new ways to compute quantities like (3), obtaining new insights on these invariants. This is indeed what happened in the nineties, leading to an important breakthrough in 1994 when Seiberg and Witten calculated (3) in a different way and pointed out the relation of Donaldson invariants to new integer invariants that nowadays carry their names.

The localization argument that led to the interpretation of (3) as Donaldson invariants is valid because the theory under consideration is exact in the weak coupling limit. Actually, one can easily argue that the topological theory under consideration is independent of the coupling constant and

thus calculations in the strong coupling limit are also exact. These type of calculations were out of the question before 1994. The situation changed dramatically after the work by Seiberg and Witten in which  $\mathcal{N} = 2$  super Yang-Mills theory was solved in the strong coupling limit [10]. Its application to the corresponding twisted version was immediate and it turned out that Donaldson invariants can be written in terms of new integer invariants now known as Seiberg-Witten invariants [11]. The development has a strong resemblance with the one described above for topological strings: certain non integer invariants can be expressed in terms of new integer invariants.

The Seiberg-Witten invariants are actually simpler to compute than Donaldson invariants. They correspond to partition functions of topological Yang-Mills theories where the gauge group is abelian. These contributions can be grouped into classes labeled by  $x = -2c_1(L)$  where  $c_1(L)$  is the first Chern class of the corresponding line bundle. The sum of contributions, each being  $\pm 1$ , for a given class  $x$  is the *integer* Seiberg-Witten invariant  $n_x$ . The strong coupling analysis of topological Yang-Mills theory leads to the following expression for (3):

$$2^{1+\frac{1}{4}(7\chi+11\sigma)} \left( e^{\left(\frac{v^2}{2}+2\lambda\right)} \sum_x n_x e^{v \cdot x} + i^{\frac{\chi+\sigma}{4}} e^{\left(-\frac{v^2}{2}-2\lambda\right)} \sum_x n_x e^{-iv \cdot x} \right). \quad (4)$$

where  $v = \sum_a \alpha_a \Sigma_a$ , and  $\chi$  and  $\sigma$  are the Euler number and the signature of the manifold  $X$ . This result matches the known structure of (3) (structure theorem of Kronheimer and Mrowka [12]) and provides a meaning to its unknown quantities in terms of the new Seiberg-Witten invariants. Equation (4) is a rather remarkable prediction that has been tested in many cases. A general proof of the relation between Donaldson and Witten invariants has been proposed recently in [13]. For a review of the subject see [14].

The situation for manifolds with  $b_2^+ = 1$  involves a metric dependence and has been worked out in detail in [15]. The formulation of Donaldson invariants in field-theoretical terms has also provided a generalization of these invariants. This generalization has been carried out in several directions: *a)* the consideration of higher-rank groups [16], *b)* the coupling to matter fields after twisting  $\mathcal{N} = 2$  hypermultiplets [17], *c)* the twist of theories involving  $\mathcal{N} = 4$  supersymmetry for gauge group  $SU(2)$  [18], and for higher rank groups [19].



### 3 Chern-Simons gauge theory

In this section I will briefly review the most significant topological quantum field theory in three dimensions. Chern-Simons gauge theory is a topological quantum field theory whose action is built out of a Chern-Simons term involving as gauge field a gauge connection associated to a group  $G$  on a three-manifold  $M$ ,

$$S = \frac{k}{4\pi} \int_M \text{Tr} \left( A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right), \quad (5)$$

where  $k$  is an integer. The natural physical observables of the theory are Wilson loops:

$$W_R^K(A) = \text{Tr}_R \text{P} \exp \oint_K A, \quad (6)$$

where  $K$  is a loop and  $R$  a representation of the gauge group. The vacuum expectation values of products of these operators are topological invariants which are related to quantum-group invariants. Given a link  $\mathcal{L}$  of  $L$  components,  $K_1, K_2, \dots, K_L$ , one computes correlators of the form  $\langle W_{R_1}^{K_1} \cdots W_{R_L}^{K_L} \rangle$ , where  $R_1, R_2, \dots, R_L$  are representations associated to each component. These quantities turn out to be polynomials in  $q = e^{\frac{2\pi i}{k+N}}$  and  $\lambda = q^N$  with integer coefficients. In what follows I will concentrate on the case of knots, denoting the vacuum expectation value of the Wilson loop (6) simply by  $W_R$ .

The observation that these correlators lead to knot and link invariants was done by Witten in 1988 [20]. He provided a completely new point of view for these invariants which turned out to be very fruitful over the years. The first study of this theory used non-perturbative field theoretical methods and led to the identification of its correlators with quantum-group invariants for knots and links [21]. Again, as in the two cases discussed in the previous section, the different methods of field theory were available to study the theory from different perspectives. As it will be now briefly reviewed, the progress made in this case is even more impressive than in the previous cases.

Besides non-perturbative methods, perturbative ones can be also applied. These were soon developed for Chern-Simons gauge theory [22], and they provided important representations of Vassiliev invariants. These invariants, proposed by Vassiliev in 1990 [23], turned out to be the coefficients of the perturbative series expansion of the correlators of Chern-Simons gauge theory [24]. Perturbative studies can be carried out in different gauges, originating

a variety of new representations of Vassiliev invariants. Among the more relevant results related to these topics are the integral expressions for Vassiliev invariants by Kontsevich [25] and by Bott and Taubes [26], as well as the recent combinatorial ones [27] in the spirit of [28]. I will not describe these developments here but refer the interested reader to the recent review [29]. In this paper I will concentrate in the new perspective emerged after studying the large  $N$  expansion of the theory. I will restrict the discussion to the case of knots on  $S^3$  with gauge group  $SU(N)$ .

Gauge theories with gauge group  $SU(N)$  admit, besides the perturbative expansion, a large- $N$  expansion. In this expansion correlators are expanded in powers of  $1/N$  while keeping the 't Hooft coupling  $t = Nx$  fixed, being  $x$  the coupling constant of the gauge theory. For example, for the free energy of the theory one has the general form,

$$F = \sum_{\substack{g \geq 0 \\ h \geq 1}}^{\infty} C_{g,h} N^{2-2g} t^{2g-2+h}. \quad (7)$$

In the case of Chern-Simons gauge theory, the coupling constant is  $x = \frac{2\pi i}{k+N}$  after taking into account the shift in  $k$  [30]. The large- $N$  expansion (7) resembles a string theory expansion and indeed the quantities  $C_{g,h}$  can be identified with the partition function of a topological open string with  $g$  handles and  $h$  boundaries, with  $N$  D-branes on  $S^3$  in an ambient six-dimensional target space  $T^*S^3$ . This was pointed out by Witten in 1992 [31]. The result makes a connection between a topological three-dimensional field theory and the topological strings described in the previous section.

In 1998 an important breakthrough took place which provided a new approach to compute quantities like (7). Using arguments inspired by the AdS/CFT correspondence (see [32] for a review), Gopakumar and Vafa [33] provided a closed string theory interpretation of the partition function (7). They conjecture that the free energy  $F$  can be expressed as,

$$F = \sum_{g \geq 0}^{\infty} N^{2-2g} F_g(t), \quad (8)$$

where  $F_g(t)$  correspond to the partition function of a topological closed string theory on the non-compact Calabi-Yau manifold  $X$  called the resolved conifold,  $\mathcal{O}(-1) \oplus \mathcal{O}(-1) \rightarrow \mathbf{P}^1$ , being  $t$  the flux of the  $B$ -field through  $\mathbf{P}^1$ . The quantities  $F_g(t)$  have been computed using physical [33] and mathematical arguments [34], proving the conjecture.

Once a new picture for the partition function of Chern-Simons gauge theory is available one should ask about the form that the expectation values of Wilson loops could take in the new context. The question was faced by Ooguri and Vafa and they provided the answer [35], later refined in [36]. The outcome is an entirely new point of view in the theory of knot and link invariants. The new picture provides a geometrical interpretation of the integer coefficients of the quantum group invariants, an issue that has been investigated during many years. To present an account of these developments one needs to review first some basic facts of large- $N$  expansions.

To consider the presence of Wilson loops it is convenient to introduce a particular generating functional. First, one performs a change basis from representations  $R$  to conjugacy classes  $C(\vec{k})$  of the symmetric group, labeled by vectors  $\vec{k} = (k_1, k_2, \dots)$  with  $k_i \geq 0$ , and  $|\vec{k}| = \sum_j k_j > 0$ . The change of basis is  $W_{\vec{k}} = \sum_R \chi_R(C(\vec{k})) W_R$ , where  $\chi_R$  are characters of the permutation group  $S_\ell$  of  $\ell = \sum_j j k_j$  elements ( $\ell$  is also the number of boxes of the Young tableau associated to  $R$ ). Second, one introduces the generating functional:

$$F(V) = \log Z(V) = \sum_{\vec{k}} \frac{|C(\vec{k})|}{\ell!} W_{\vec{k}}^{(c)} \Upsilon_{\vec{k}}(V), \quad (9)$$

where  $Z(V) = \sum_{\vec{k}} \frac{|C(\vec{k})|}{\ell!} W_{\vec{k}} \Upsilon_{\vec{k}}(V)$  and  $\Upsilon_{\vec{k}}(V) = \prod_j (\text{Tr} V^j)^{k_j}$ . In these expressions  $|C(\vec{k})|$  denotes the number of elements of the class  $C(\vec{k})$  in  $S_\ell$ . The reason behind the introduction of this generating functional is that the large- $N$  structure of the connected Wilson loops,  $W_{\vec{k}}^{(c)}$ , turns out to be very simple:

$$\frac{|C(\vec{k})|}{\ell!} W_{\vec{k}}^{(c)} = \sum_{g=0}^{\infty} x^{2g-2+|\vec{k}|} F_{g,\vec{k}}(\lambda), \quad (10)$$

where  $\lambda = e^t$  and  $t = Nx$  is the 't Hooft coupling. Writing  $x = t/N$ , it corresponds to a power series expansion in  $1/N$ . As before, the expansion looks like a perturbative series in string theory where  $g$  is the genus and  $|\vec{k}|$  is the number of holes. Ooguri and Vafa conjectured in 1999 the appropriate string theory description of (10). It correspond to an open topological string theory (notice that the ones described in the previous section were closed) whose target space is the resolved conifold  $X$ . The contribution from this theory will lead to open-string analogs of Gromov-Witten invariants.

In order to describe in more detail the fact that one is dealing with open strings, some new data needs to be introduced. Here is where the knot

intrinsic to the Wilson loop enters. Given a knot  $K$  on  $S^3$ , let us associate to it a Lagrangian submanifold  $C_K$  with  $b_1 = 1$  in the resolved conifold  $X$  and consider a topological open string on it. The contributions in this open topological string are localized on holomorphic maps  $f : \Sigma_{g,h} \rightarrow X$  with  $h = |\vec{k}|$  which satisfy:  $f_*[\Sigma_{g,h}] = \mathcal{Q}$ , and  $f_*[C] = j[\gamma]$  for  $k_j$  oriented circles  $C$ . In these expressions  $\gamma \in H_1(C_K, \mathbf{Z})$ , and  $\mathcal{Q} \in H_2(X, C_K, \mathbf{Z})$ , *i.e.*, the map is such that  $k_j$  boundaries of  $\Sigma_{g,h}$  wrap the knot  $j$  times, and  $\Sigma_{g,h}$  itself gets mapped to a relative two-homology class characterized by the Lagrangian submanifold  $C_K$ . The number of these maps (in the sense described in the previous section) constitute the open-string analogs of Gromov-Witten invariants. They will be denoted by  $N_{g,\vec{k}}^{\mathcal{Q}}$ . Comparing to the situation that led to (1) in the closed string case one concludes that in this case the quantities  $F_{g,\vec{k}}(\lambda)$  in (10) must take the form:

$$F_{g,\vec{k}}(\lambda) = \sum_{\mathcal{Q}} N_{g,\vec{k}}^{\mathcal{Q}} e^{\int_{\mathcal{Q}} \omega}, \quad t = \int_{\mathbf{P}^1} \omega, \quad (11)$$

where  $\omega$  is the Kähler class of the Calabi-Yau manifold  $X$  and  $\lambda = e^t$ . For any  $\mathcal{Q}$ , one can always write  $\int_{\mathcal{Q}} \omega = Qt$  where  $Q$  is in general a half-integer number. Therefore,  $F_{g,\vec{k}}(\lambda)$  is a polynomial in  $\lambda^{\pm \frac{1}{2}}$  with rational coefficients.

The result (11) is very impressive but still does not provide a representation where one can assign a geometrical interpretation to the integer coefficients of the quantum-group invariants. Notice that to match a polynomial invariant to (11), after obtaining its connected part, one must expand it in  $x$  after setting  $q = e^x$  keeping  $\lambda$  fixed. One would like to have a refined version of (11), in the spirit of what was described in the previous section leading from the Gromov-Witten invariants  $N_g^\beta$  of (1) to the new integer invariants  $n_g^\beta$  of (2). This study was indeed done in [35] and later improved in [36]. The outcome is that, indeed,  $F(V)$  can be expressed in terms of integer invariants in complete analogy with the description presented in the previous section for topological strings.

In order to present these results one needs to perform first a reformulation of the quantum-group invariants or vacuum expectation values of Wilson loops. Instead of considering  $W_R(q, \lambda)$ , a corrected version of it,  $f_R(q, \lambda)$ , will be studied. These reformulated polynomial invariants have the form:

$$f_R(q, \lambda) = \sum_{d,m=1}^{\infty} (-1)^{m-1} \frac{\mu(d)}{dm} \sum_{\{\vec{k}^{(j)}, R_j\}} \chi_R \left( C \left( \left( \sum_{j=1}^m \vec{k}^{(j)} \right)_d \right) \right)$$

$$\times \prod_{j=1}^m \frac{|C(\vec{k}^{(j)})|}{\ell_j!} \chi_{R_j}(C(\vec{k}^{(j)})) W_{R_j}(q^d, \lambda^d), \quad (12)$$

where  $(\vec{k}_d)_{di} = k_i$  and zero otherwise. In this expression  $\mu(d)$  is the Moebius function. In spite of its frightening form, the reformulated polynomial invariants  $f_R(q, \lambda)$  are just quantum-group invariants  $W_R(q, \lambda)$  plus lower order terms, understanding by this terms which contain quantum-group invariants carrying representations whose associated Young tableaux have a lower number of boxes. For example, for the simplest cases:

$$\begin{aligned} f_{\square}(q, \lambda) &= W_{\square}(q, \lambda), \\ f_{\blacksquare}(q, \lambda) &= W_{\blacksquare}(q, \lambda) - \frac{1}{2}(W_{\square}(q, \lambda)^2 + W_{\square}(q^2, \lambda^2)), \\ f_{\boxminus}(q, \lambda) &= W_{\boxminus}(q, \lambda) - \frac{1}{2}(W_{\square}(q, \lambda)^2 - W_{\square}(q^2, \lambda^2)). \end{aligned} \quad (13)$$

The nice feature of the reformulated quantities is that  $F(V)$  acquires a very simple form in terms of them:

$$F(V) = \sum_{d=1}^{\infty} \sum_R \frac{1}{d} f_R(q^d, \lambda^d) \text{Tr} V^d, \quad (14)$$

and therefore they seem to be the right quantities to express an alternative point of view in which the embedded Riemann surfaces can be regarded as D-branes (recall equation (2)).

To present the conjectured form of the reformulated  $f_R(q, \lambda)$  a couple of new ingredients are needed. One needs the Clebsch-Gordon coefficients  $C_{R,R',R''}$  of the symmetric group (they satisfy  $V_R \otimes V_{R'} = \sum_{R''} C_{R,R',R''} V_{R''}$ ), and monomials  $S_R(q)$  defined as follows:  $S_R(q) = (-1)^d q^{d - \frac{\ell-1}{2}}$  if  $R$  is a hook representation, with  $\ell - d$  boxes in the first row, and  $S_R(q) = 0$  otherwise. The conjecture presented in [36] states that the reformulated invariants have the form:

$$f_R(q, \lambda) = \sum_{g \geq 0} \sum_{Q, R', R''} C_{RR'R''} N_{R',g,Q} S_{R''}(q) (q^{\frac{1}{2}} - q^{-\frac{1}{2}})^{2g-1} \lambda^Q, \quad (15)$$

where  $N_{R,g,Q}$  are *integer* invariants which posses a geometric interpretation. These quantities are the analog of the integers  $n_g^\beta$  in Gromov-Witten theory. They can be described in terms of the moduli space of Riemann surfaces

with boundaries embedded into a Calabi-Yau manifold. Their geometrical interpretation has been treated recently in [37].

The structure (15) has been verified for a variety of non-trivial knots and links, and representations up to four boxes [38, 36]. For the unknot, the whole picture have been verified in complete detail [35, 39]. The form of  $F(V)$  for the unknot can be easily computed in Chern-Simons gauge theory,

$$F(V) = \sum_{d=1}^{\infty} \frac{\lambda^{\frac{d}{2}} - \lambda^{-\frac{d}{2}}}{2d \sin(\frac{dx}{2})} \text{Tr}_{\square} V^d, \quad (16)$$

leading to an expansion (10) of the form:

$$F_{g,(0,\dots,0,1,0,\dots,0)}(\lambda) = \frac{(1 - 2^{1-2g})|B_{2g}|}{(2g)!} d^{2g-2} (\lambda^{\frac{d}{2}} - \lambda^{-\frac{d}{2}}), \quad (17)$$

where the 1 in  $F_{g,(0,\dots,0,1,0,\dots,0)}$  is located in the  $d^{\text{th}}$  position. Form these equations one can easily read the numbers which correspond to the open-string analogs of Gromov-Witten invariants,  $N_{g,\vec{k}}^{\mathcal{Q}}$ , in (11), as well as the new integer invariants present in the general expression (15):

$$N_{\square,0,\frac{1}{2}} = -N_{\square,0,-\frac{1}{2}} = 1. \quad (18)$$

In this section I have shown how the string theory description of the large- $N$  expansion of Chern-Simons gauge theory provides a new point of view in the study of knot and link invariants. Quantum-group polynomial invariants are reformulated so that their integer coefficients and exponents possess a geometric interpretation. The new integer invariants are identified in terms of topological properties of the moduli space of Riemann surfaces with boundaries embedded into a Calabi-Yau manifold. The new framework provides strong predictions on the algebraic structure of the quantum group polynomial invariants.

Many open problems remain to be studied. One would like to know what are the implications of the skein rules on the new invariants, and, on the contrary, one would like to understand the new algebraic structure of the polynomial invariants from the point of view of quantum groups. Also, one should study the extension of the formalism to other gauge groups. From a more mathematical perspective, a detailed analysis of the moduli spaces involved in the computation of the new invariants is needed, as well as the

development of computational technics. One should also analyze the picture emerging from the theories which are mirror to the topological strings involved. Some work in this direction has been done recently in [40].

In this paper I have described how the many faces of quantum field theory and string theory provide a variety of important insights in a selected set of problems in topology. It is particularly remarkable the connection between the first and the third contexts considered. One observes a fascinating interplay between string theory, knot theory and enumerative geometry which opens new fields of study.

### **Acknowledgments**

I would like to thank the organizers of the special seminar in honor of F. J. Ynduráin for inviting me to deliver a talk at the meeting held at Sitges, Spain, on February 9, 2001, in the context of the XXIX International Meeting on Fundamental Physics. I would like to thank also M. Alvarez, C. Lozano, P.M. Llatas, E. Pérez, A. V. Ramallo and, specially, M. Mariño, for collaborations and discussions on many of the topics described here. This work is supported in part by Ministerio de Ciencia y Tecnología under grant PB96-0960, and by Xunta de Galicia under grant PGIDT00-PXI-20609.

## Afterword

This paper is dedicated to Francisco J. Ynduráin, Paco, as his friends use to call him. Paco was my first quantum field theory teacher in the Universidad Autónoma de Madrid. I learnt from him the basics of the subject, a topic that he mastered and enjoyed. He is an excellent teacher, emphasizing the essentials and avoiding cumbersome detours. These features are clear in his textbooks, pieces of great work already enjoyed by several generations. I also started with Paco my career as a researcher. I wrote with him my first paper, dealing with some aspects of proton decay. That paper contains the only measurable physics prediction in which I has been involved. Unfortunately, it has been ruled out. I have made, however, testable mathematical predictions that have been verified (the risk is much lower). As a young researcher, I learnt from Paco that one should always try to face important problems no matter how hard or fashionable they are. Paco has certainly made many valuable contributions to theoretical physics. I am sure that he will continue doing so for many years. He has also made important efforts to promote high-energy physics in Spain (both, theoretical and experimental), to rise it to the level that we enjoy nowadays. In this task, he was always guided by high-quality standards. We should all thank him for his firm commitment.



## References

- [1] E. Witten, “Supersymmetry and Morse theory”, *J. Diff. Geom.* **17** (1982) 661.
- [2] A. Floer, “Morse theory for fixed points of symplectic diffeomorphisms”, *Bull. AMS* **16** (1987) 279.
- [3] M. F. Atiyah, “New invariants of three and four dimensional manifolds”, in *The Mathematical Heritage of Herman Weyl, Proc. Symp. Pure Math.* AMS **48** (1988) 285-299.
- [4] S. K. Donaldson, “Polynomial invariants for smooth four-manifolds”, *Topology* **29** (1990) 257.
- [5] E. Witten, “Topological quantum field theory”, *Commun. Math. Phys.* **117** (1988) 353.
- [6] E. Witten, “Topological sigma models”, *Commun. Math. Phys.* **118** (1988) 411.
- [7] J.M.F. Labastida and P.M. Llatas, “Topological matter in two dimensions”, hep-th/9112051, *Nucl. Phys.* **B379** (1992) 220; E. Witten, “Mirror manifolds and topological field theory”, hep-th/9112056, *Mirror symmetry I*, Yau, S.T. (ed.), 1992, 121.
- [8] R. Gopakumar and C. Vafa, “M-Theory and Topological Strings I, II”, hep-th/9809187, hep-th/9812127.
- [9] C. Faber and R. Pandharinpande, “Hodge integrals and Gromov-Witten theory”, math.AG/9810173; R. Pandharinpande, “Hodge integrals and degenerate contributions”, math.AG/9811140; S. Katz, A. Klemm and C. Vafa, “M-Theory, Topological Strings and Spinning Black Holes”, hep-th/99101181.
- [10] N. Seiberg and E. Witten, “Electric-magnetic duality, monopole condensation, and confinement in  $\mathcal{N} = 2$  supersymmetric Yang-Mills theory”, hep-th/9407087, *Nucl. Phys.* **B426** (1994) 19.
- [11] E. Witten, “Monopoles and four-manifolds”, hep-th/9411102, *Math. Res. Lett.* **1** (1994) 769.

- [12] P.B. Kronheimer and T.S. Mrowka, “Recurrence relations and asymptotics for four-manifold invariants”, *Bull. Am. Math. Soc.* **30** (1994) 215; “Embedded surfaces and the structure of Donaldson’s polynomial invariants”, *J. Diff. Geom.* **41** (1995) 573.
- [13] Paul M. N. Feehan and Thomas G. Leness, “SO(3) Monopoles, Level-One Seiberg-Witten Moduli Spaces, and Witten’s Conjecture in Low Degrees”, math.DG/0106238.
- [14] J.M.F. Labastida and C. Lozano, “Lectures in topological quantum field theory,” hep-th/9709192, *Trends in Theoretical Physics*, H. Falomir, R. Gamboa, F. Schaposnik, eds. AIP, New York, 1998, CP 419, 54.
- [15] G. Moore and E. Witten, “Integrating over the  $u$ -plane in Donaldson theory,” hep-th/9709193, *Adv. Theor. Math. Phys.* **1** (1998) 298.
- [16] M. Mariño and G. Moore, “The Donaldson-Witten function for gauge groups of rank larger than one,” hep-th/9802185, *Commun. Math. Phys.* **199** (1999) 25.
- [17] J.M.F. Labastida and M. Mariño, “A topological Lagrangian for monopoles on four-manifolds”, hep-th/9503105, *Phys. Lett.* **B351** (1995) 146; “Non-abelian monopoles on four-manifolds”, hep-th/9504010, *Nucl. Phys.* **B448** (1995) 373; “Twisted  $\mathcal{N} = 2$  supersymmetry with central charge and equivariant cohomology”, hep-th/9603169, *Comm. Math. Phys.* **185** (1997) 37.
- [18] C. Vafa and E. Witten, “A strong coupling test of  $S$ -duality, hep-th/9408074, *Nucl. Phys.* **B431** (1994) 3.
- [19] J.M.F. Labastida and C. Lozano, “The Vafa-Witten Theory for Gauge Group SU(N)” hep-th/9709192, *Adv. Theor. Math. Phys.* **5** (1999) 1201.
- [20] E. Witten, “Quantum field theory and the Jones polynomial”, *Commun. Math. Phys.* **121** (1989) 351.
- [21] V.F.R. Jones, “Hecke algebras representations of braid groups and link polynomials”, *Ann. of Math.* **126** (1987) 335; V.G. Turaev, “The Yang-Baxter equation and invariants of links”, *Inv. Math.* **92** (1988) 527.

- [22] E. Guadagnini, M. Martellini and M. Mintchev, “Wilson lines in Chern-Simons theory and link invariants”, *Nucl. Phys.* **B330** (1990) 575; D. Bar-Natan, “Perturbative aspects of Chern-Simons topological quantum field theory”, Ph.D. Thesis, Princeton University, 1991.
- [23] V. A. Vassiliev, “Cohomology of knot spaces”, *Theory of singularities and its applications, Advances in Soviet Mathematics* AMS **1** 1990, 23-69.
- [24] J.S. Birman, “New points of view in knot theory”, *Bull. Amer. Math. Soc.* **28** (1993) 253; D. Bar-Natan, “On the Vassiliev knot invariants”, *Topology* **34** (1995) 423; J.S. Birman and X.S. Lin, “Knot polynomials and Vassiliev invariants”, *Inv. Math.* **111** (1993) 225; J.M.F. Labastida and E. Pérez, “Gauge-invariant operators for singular knots in Chern-Simons gauge theory”, hep-th/9712139, *Nucl. Phys.* **B527** (1998) 499.
- [25] M. Kontsevich, “Vassiliev knot invariants”, *Advances in Soviet Math.* **16**, Part 2 (1993) 137.
- [26] R. Bott and C. Taubes, “On the self-linking of knots”, *Jour. Math. Phys.* **35** (1994) 5247.
- [27] J.M.F. Labastida and E. Pérez, “Combinatorial formulae for Vassiliev invariants from Chern-Simons gauge theory”, hep-th/9712139, *J. Math. Phys.* **41** (2000) 2658.
- [28] M. Goussarov, M. Polyak and O. Viro, “Finite type invariants of classical and virtual knots”, *Int. Math. Res. Notices* **11** (1994) 445.
- [29] J.M.F. Labastida, “Chern-Simons Gauge Theory: Ten Years After”, hep-th/9905057, *Trends in Theoretical Physics II*, H. Falomir, R. Gamboa, F. Schaposnik, eds. AIP, New York, 1999, CP 484, 1.
- [30] L. Alvarez-Gaumé, J.M.F. Labastida and A. V. Ramallo, “A Note on Perturbative Chern-Simons Theory”, *Nucl. Phys.* **B334** (1990) 103.
- [31] E. Witten, “Chern-Simons gauge theory as a string theory”, hep-th/9207094, in *The Floer memorial volume*, H. Hofer, C.H. Taubes, A. Weinstein and E. Zehner, eds., Birkhäuser 1995, p. 637.

- [32] O. Aharony, S. Gubser, J. Maldacena, H. Ooguri and Y. Oz, “Large  $N$  field theories, string theory and gravity”, hep-th/9905111, *Phys. Rep.* **323** (2000) 183.
- [33] R. Gopakumar and C. Vafa, “On the gauge theory/geometry correspondence”, hep-th/9811131, *Adv. Theor. Math. Phys.* **3** (1999) 1415.
- [34] C. Faber and R. Pnadhari, “Hodge integrals and Gromov-Witten theory”, math.AG/9810173.
- [35] H. Ooguri and C. Vafa, “Knot invariants and topological strings”, hep-th/9912123, *Nucl. Phys.* **B577** (2000) 419.
- [36] J.M.F. Labastida, M. Mariño and C. Vafa, “Knots, links and branes at large  $N$ ”, hep-th/0010102, *J. of High Energy Physics* **0011** (2000) 007.
- [37] J.M.F. Labastida, Marcos Mariño, “A new point of view in the theory of knot and link invariants”, math.qa/0104180.
- [38] J.M.F. Labastida and M. Mariño, “Polynomial invariants for torus knots and topological strings”, hep-th/0004196, *Commun. Math. Phys.* **217** (2001) 423; P. Ramadevi and T. Sarkar, “On link invariants and topological string amplitudes,” hep-th/0009188, *Nucl. Phys.* **B600** (2001) 487.
- [39] S. Katz and M. Liu, “Enumerative geometry of stable maps with Lagrangian boundary conditions and multiple covers of the disc,” math.AG/0103074; J. Li and Y.S. Song, “Open string instantons and relative stable morphisms,” hep-th/0103100.
- [40] Mina Aganagic, Albrecht Klemm and Cumrun Vafa, “Disk instantons, mirror symmetry and the duality web”, hep-th/0105045; Mina Aganagic and Cumrun Vafa, “Mirror Symmetry and a  $G_2$  Flop”, hep-th/0105225.